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A note on a sum theorem for dimension $\mathcal{K}$-Ind

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ABSTRACT

Main results are:

1. Let Y be a closed subspace of a hereditarily normal X such that $\mathcal{K}\text{-Ind}\,Y \leqslant n$ and $\mathcal{K}\text{-Ind}(X \setminus Y) \leqslant n$. Then $\mathcal{K}\text{-Ind}\,X \leqslant n$.
2. Let X be a perfectly normal space. Then a finite sum theorem for dimension $\mathcal{K}$-Ind holds in X if and only if $\mathcal{K}$-Ind is monotonic in X.

We denote by $\mathcal{K}$ a non-empty set of finite complete simplicial complexes.

1. Introduction

In [4] inductive dimension function $\mathcal{K}$-Ind, where $\mathcal{K}$ is a non-empty set of finite simplicial complexes, was introduced (look at Definition 2.4). This dimension is an extension of the classical inductive dimension Ind, since $\{0,1\}\text{-Ind}\,X = \text{Ind}\,X$ for every normal space. Generally,

$$\mathcal{K}\text{-Ind}\,X \leqslant \text{Ind}\,X.$$

In [4] it was proved that

$$\mathcal{K}\text{-Ind}\,X = \text{Ind}\,X$$

if and only if $\mathcal{K}$ contains a disconnected complex K.

One of the main questions concerning dimension $\mathcal{K}$-Ind is the following one:

Let a perfectly normal space X be the union of its closed subspaces X_i, $i = 1, 2 \ldots$. Is it true that

$$\mathcal{K}\text{-Ind}\,X = \sup\{\mathcal{K}\text{-Ind}\,X_i\colon\ i = 1, 2, \ldots\}?$$

The answer is unknown even if $X = X_1 \cup X_2$.

Here we prove (Theorem 3.4) that the finite sum theorem for dimension $\mathcal{K}$-Ind holds for subspaces of a perfectly normal space X if and only if dimension $\mathcal{K}$-Ind is monotonic in subspaces of X. The proof is based on the following

Finite Dowker theorem *(Theorem 3.1). Let Y be a closed subspace of a hereditarily normal space X such that $\mathcal{K}\text{-Ind}\,Y \leqslant n$, $\mathcal{K}\text{-Ind}(X \setminus Y) \leqslant n$. Then $\mathcal{K}\text{-Ind}\,X \leqslant n$.*

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Let us recall that C.H. Dowker proved (look at [1,2]) the following

Theorem D. *Let a hereditarily normal space X be the union of its subspaces X_i, $i = 1, 2, \dots,$ such that* $\operatorname{Ind} X_i \leqslant n$, *$i = 1, 2 \dots,$ and $\bigcup\{X_i\colon\ i = 1, 2, \dots, k\}$ is closed for $k = 1, 2, \dots$. Then* $\operatorname{Ind} X \leqslant n$.

Theorem 3.1 implies a finite version of Theorem D for dimension $\mathcal{K}$-Ind.

2. Preliminaries

2.1. In what follows $\mathcal{K}$ stands for a non-empty set of finite complete simplicial complexes K, which we call *complexes*. For a complex K by $v(K)$ we denote the set of all its vertices. A simplicial complex, which is the *nerve* of a finite family $\alpha = \{A_1, \dots, A_s\}$ of sets, is denoted by $N(\alpha)$.

By a *space* we mean a topological normal T_1-space. For a space X by $\exp X$ we denote the set of all closed subsets of X. By $\mathrm{Fin}_s(\exp X)$ we denote the set of all finite sequences $\Phi = (F_1, \dots, F_m)$, $F_j \in \exp X$, $j = 1, \dots, m$.

Definition 2.2. ([3]) Let X be a space, K be a complex, and $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$. A sequence $u = (U_1, \dots, U_s)$, $s \geqslant m$, of open subsets of X is said to be a *K-neighbourhood* of Φ if $F_j \subset U_j$, $j = 1, \dots, m$, and there is an embedding $N(u) \subset K$. One can number vertices $a_j \in v(K)$ so that the embedding $N(u) \subset K$ is defined by the correspondence $U_j \to a_j$.

Definition 2.3. ([3]) A set $P \subset X$ is called a *K-partition* of $\Phi \in \mathrm{Fin}_s(\exp X)$ (notation: $P \in \mathrm{Part}(\Phi, K)$) if $P = X \setminus \bigcup u$, where u is a K-neighbourhood of Φ.

If a K-partition of Φ exists, then $N(\Phi) \subset K$. Put

$$\mathrm{Exp}_K(X) = \bigl\{\Phi \in \mathrm{Fin}_s(\exp X)\colon\ N(\Phi) \subset K\bigr\}. \tag{2.1}$$

Definition 2.4. ([4]) To every space X one assigns the dimension $\mathcal{K}\text{-}\operatorname{Ind} X$ which is an integer $\geqslant -1$ or ∞. The dimension function $\mathcal{K}$-Ind is defined in the following way:

(1) $\mathcal{K}\text{-}\operatorname{Ind} X = -1 \iff X = \emptyset$;
(2) $\mathcal{K}\text{-}\operatorname{Ind} X \leqslant n$, where $n = 0, 1, \dots,$ if for every $K \in \mathcal{K}$ and $\Phi \in \mathrm{Exp}_K(X)$ there exists a partition $P \in \mathrm{Part}(\Phi, K)$ such that $\mathcal{K}\text{-}\operatorname{Ind} P \leqslant n - 1$;
(3) $\mathcal{K}\text{-}\operatorname{Ind} X = \infty$, if $\mathcal{K}\text{-}\operatorname{Ind} X > n$ for all $n \geqslant -1$.

If the set $\mathcal{K}$ contains only one complex K, we write $\mathcal{K} = K$ and $\mathcal{K}\text{-}\operatorname{Ind} X = K\text{-}\operatorname{Ind} X$.

Theorem 2.5. *For every space X,* $\{0, 1\}\text{-}\operatorname{Ind} X = \operatorname{Ind} X$. □

Theorem 2.6. *([4]) If Y is a closed subspace of a space X, then* $\mathcal{K}\text{-}\operatorname{Ind} Y \leqslant \mathcal{K}\text{-}\operatorname{Ind} X$. □

Theorem 2.7. *([4]) If $X = \bigoplus\{X_\alpha\colon\ \alpha \in A\}$ is a discrete union of spaces X_α, then*

$$\mathcal{K}\text{-}\operatorname{Ind} X = \sup\{\mathcal{K}\text{-}\operatorname{Ind} X_\alpha\colon\ \alpha \in A\}. \qquad \square$$

Lemma 2.8. *If U is an open F_σ-subset of a space X, then U is a cozero-set, i.e. there exists a continuous function $\varphi : X \to [0, 1]$ such that $U = \varphi^{-1}(0, 1]$.* □

Strong swelling lemma 2.9. *Let $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$. Then there exists a family $u = (U_1, \dots, U_m)$ of open subsets of X such that $F_j \subset U_j$, $j = 1, \dots, m$, and $N(\mathrm{Cl}(u)) = N(\Phi)$, where $\mathrm{Cl}(u) = (\mathrm{Cl}(U_1), \dots, \mathrm{Cl}(U_m))$.* □

Nerve lemma 2.10. *([4]) Let Y be subspace of a space X, $\alpha = (A_1, \dots, A_m)$ be a sequence of subsets of X, and $\beta = (B_1, \dots, B_m)$ be a sequence of subsets of Y such that $N(\alpha), N(\beta) \subset K$ and $A_j \cap Y \subset B_j$, $j = 1, \dots, m$. Let $C_j = A_j \cup B_j$ and $\gamma = (C_1, \dots, C_m)$. Then $N(\gamma) \subset K$.* □

3. Main results

Let $\mathcal{K}$ be a non-empty set of complexes and let X be a hereditarily normal space.

Theorem 3.1. *Let Y be a closed subspace of a hereditarily normal space X such that* $\mathcal{K}\text{-}\operatorname{Ind} Y \leqslant n$, $\mathcal{K}\text{-}\operatorname{Ind}(X \setminus Y) \leqslant n$. *Then* $\mathcal{K}\text{-}\operatorname{Ind} X \leqslant n$.

Proof. We shall apply induction with respect to n. For $n=-1$ the theorem is obvious. Assume that the corresponding statements hold for dimensions less than $n \geqslant 0$ and consider a hereditarily normal space X satisfying the assumption of our theorem. Let $K \in \mathcal{K}$, $\Phi=(F_1,\dots,F_m) \in \mathrm{Exp}_K(X)$, and $\Phi|Y=(F_1\cap Y,\dots,F_m\cap Y)$. Then $\Phi|Y \in \mathrm{Exp}_K(Y)$. Since $\mathcal{K}\text{-Ind}\,Y \leqslant n$, there is a family $u=(U_1,\dots,U_m)$ of open subsets of Y such that

$$F_j \cap Y \subset U_j, \quad j=1,\dots,m; \tag{3.1}$$

$$N(u) \subset K; \tag{3.2}$$

$$\mathcal{K}\text{-Ind}(Y \setminus U_1 \cup \dots \cup U_m) \leqslant n-1. \tag{3.3}$$

Put $P=Y\setminus U_1\cup\dots\cup U_m$ and $Z=X\setminus P$. The family u is an open cover of a normal space $Y\setminus P$. Hence there exist closed subsets A_j of a space $Y\setminus P$ such that

$$F_j \cap Y \subset A_j \subset U_j; \tag{3.4}$$

$$A_1 \cup \dots \cup A_m = Y \setminus P. \tag{3.5}$$

Put $\alpha=(A_1,\dots,A_m)$. From (3.2) and (3.4) it follows that

$$N(\alpha) \subset K. \tag{3.6}$$

Since $Y\setminus P$ is closed in Z, the sets $B_j=A_j\cup F_j$, $j=1,\dots,m$, are closed in Z. Put $\beta=(B_1,\dots,B_m)$. The condition $\Phi \in \mathrm{Exp}_K(X)$ is equivalent to

$$N(\Phi) \subset K. \tag{3.7}$$

From (3.6), (3.7) and the Nerve lemma (Lemma 2.10) it follows that

$$N(\beta) \subset K. \tag{3.8}$$

Consequently, according to the Strong swelling lemma (Lemma 2.9) there exists a family $v=(V_1,\dots,V_m)$ of open subsets of Z such that

$$B_j \subset V_j, \quad j=1,\dots,m; \tag{3.9}$$

$$N(\delta)=N(\beta) \subset K, \tag{3.10}$$

where $\delta=(D_1,\dots,D_m)$ and $D_j=\mathrm{Cl}_Z(V_j)$.

Put $E_j=D_j\setminus Y$, $j=1,\dots,m$, and $\epsilon=(E_1,\dots,E_m)$. The sets E_j are closed in $X\setminus Y$ and

$$N(\epsilon) \subset K \tag{3.11}$$

according to (3.10). But $\mathcal{K}\text{-Ind}(X\setminus Y)\leqslant n$. Consequently, according to (3.11) there exists a family $w=(W_1,\dots,W_m)$ of open subsets of $X\setminus Y$ such that

$$E_j \subset W_j, \quad j=1,\dots,m; \tag{3.12}$$

$$N(w) \subset K; \tag{3.13}$$

$$\mathcal{K}\text{-Ind}\,Q \leqslant n-1, \tag{3.14}$$

where

$$Q = X\setminus Y \cup W_1 \cup \dots \cup W_m. \tag{3.15}$$

Put $G_j=V_j\cup W_j$, $j=1,\dots,m$, and $\gamma=(G_1,\dots,G_m)$.

From (3.10), (3.13), and the Nerve lemma (Lemma 2.10) it follows that

$$N(\gamma) \subset K. \tag{3.16}$$

Condition (3.9) implies that

$$F_j \subset G_j, \quad j=1,\dots,m. \tag{3.17}$$

Consequently, γ is a K-neighbourhood of Φ in X. Then the set

$$R = X \setminus G_1 \cup \dots \cup G_m \tag{3.18}$$

is a K-partition of Φ in X. We claim that

$$R = P \cup Q. \tag{3.19}$$

To check (3.19) it suffices to prove that

$$R \cap Y = P; \tag{3.20}$$

$$R \cap (X \setminus Y) = Q. \tag{3.21}$$

From definition of G_j it follows that

$$G_j \cap Y = V_j \cap Y. \tag{3.22}$$

Hence

$$(G_! \cup \cdots \cup G_m) \cap Y = (V_1 \cup \cdots \cup V_m) \cap Y.$$

Condition (3.22) is equivalent to

$$Y \setminus G_1 \cup \cdots \cup G_m = Y \setminus V_1 \cup \cdots \cup V_m. \tag{3.23}$$

From (3.18) it follows that

$$Y \setminus G_1 \cup \cdots \cup G_m = R \cap Y. \tag{3.24}$$

On the other hand, $V_j \subset Z$ implies that $(V_1 \cup \cdots \cup V_m) \cap P = \emptyset$. Consequently, $P \subset Y \setminus V_1 \cup \cdots \cup V_m \subset$ (in view of (3.9)) $\subset Y \setminus B_1 \cup \cdots \cup B_m \subset$ (because of $B_j = A_j \cup F_j$) $\subset Y \setminus A_1 \cup \cdots \cup A_m =$ (in accordance with (3.5)) $= P$.

Hence

$$P = Y \setminus V_1 \cup \cdots \cup V_m. \tag{3.25}$$

From (3.24) and (3.25) we get (3.20).

The definition of E_j implies that

$$V_j \setminus Y \subset E_j, \quad j = 1, \ldots, m. \tag{3.26}$$

Conditions (3.12) and (3.26) yield

$$V_j \setminus Y \subset W_j, \quad j = 1, \ldots, m. \tag{3.27}$$

Consequently, the definition of G_j implies that

$$G_j \cap (X \setminus Y) = W_j. \tag{3.28}$$

Then $R \cap (X \setminus Y) =$ (according to (3.18)) $= (X \setminus G_1 \cup \cdots \cup G_m) \cap (X \setminus Y) = X \setminus Y \cup G_1 \cup \cdots \cup G_m =$ (in view of (3.28)) $= X \setminus Y \cup W_1 \cup \cdots \cup W_m =$ (because of (3.15)) $= Q$.

Thus, the condition (3.21) is checked as well. Hence the equality (3.19) is proved. Since $P \cap Q = \emptyset$, we have $Q = R \setminus P$. On the other hand,

$$\mathcal{K}\text{-Ind}\, P \leqslant \big(\text{in view of (3.3)}\big) \leqslant n - 1;$$

$$\mathcal{K}\text{-Ind}\, Q \leqslant \big(\text{because of (3.14)}\big) \leqslant n - 1.$$

Consequently, by the inductive assumption we have

$$\mathcal{K}\text{-Ind}\, R \leqslant n - 1. \tag{3.29}$$

The condition (3.29) implies that $\mathcal{K}\text{-Ind}\, X \leqslant n$. □

Let us consider the following properties of a space X:

(μ_n) For each subspace $Y \subset X$ and every open subspace U of Y, if $\mathcal{K}\text{-Ind}\, Y \leqslant n$, then $\mathcal{K}\text{-Ind}\, U \leqslant n$.

(μ_n^0) For each subspace $Y \subset X$ and every open F_σ-subspace U of Y, if $\mathcal{K}\text{-Ind}\, Y \leqslant n$, then $\mathcal{K}\text{-Ind}\, U \leqslant n$.

(σ_n) For each subspace $Y \subset X$ and every pair Y_1, Y_2 of closed subspaces of Y such that $Y = Y_1 \cup Y_2$, if $\mathcal{K}\text{-Ind}\, Y_i \leqslant n$, $i = 1, 2$, then $\mathcal{K}\text{-Ind}\, Y \leqslant n$.

As a corollary of Theorem 3.1 we have

Proposition 3.2. *If a hereditarily normal space X has property (μ_n), then it also has property (σ_n).*

Proof. Consider a subspace $Y \subset X$ and a pair Y_1, Y_2 of closed subspaces of Y such that $Y = Y_1 \cup Y_2$ and $\mathcal{K}$-$\operatorname{Ind} Y_i \leqslant n$, $i = 1, 2$. By virtue of (μ_n) the set $Y \setminus Y_1$ satisfies the inequality $\mathcal{K}$-$\operatorname{Ind}(Y \setminus Y_1) \leqslant n$. Applying Theorem 3.1 to the space Y and the pair Y_1, $Y \setminus Y_1$ we obtain the inequality $\mathcal{K}$-$\operatorname{Ind} Y \leqslant n$. □

Proposition 3.3. *If a hereditarily normal space X has property (σ_n), then it also has property (μ_n^0).*

Proof. Let $Y \subset X$ and let U be an open F_σ-set in Y. Then there exists a continuous function $f : Y \to I$ such that $U = f^{-1}((0, 1])$ (see Lemma 2.8). The sets $B_i = f^{-1}([1/i + 1, 1/i])$, $i = 1, 2, \ldots$, are closed in Y. By the closed subspace theorem (Theorem 2.6) we have

$$\mathcal{K}\text{-}\operatorname{Ind} B_i \leqslant n, \quad i = 1, 2, \ldots. \tag{3.30}$$

Consider the sequences

$$B_{2i+1}, \quad i = 0, 1, 2, \ldots; \qquad B_{2i+2}, \quad i = 0, 1, 2, \ldots.$$

They are discrete. Put

$$A_1 = \bigcup \{B_{2i+1},\ i = 0, 1, 2, \ldots\}; \qquad A_2 = \bigcup \{B_{2i+2},\ i = 0, 1, 2, \ldots\}.$$

By the Discrete sum theorem (Theorem 2.7) and (3.30) we have

$$\mathcal{K}\text{-}\operatorname{Ind} A_1 \leqslant n; \qquad \mathcal{K}\text{-}\operatorname{Ind} A_2 \leqslant n.$$

But $A_1 \cup A_2 = U$. Consequently, property (σ_n) yields $\mathcal{K}$-$\operatorname{Ind} U \leqslant n$. □

Since properties (μ_n) and (μ_n^0) are equivalent in perfectly normal spaces, from Propositions 3.2 and 3.3 we get

Theorem 3.4. *Properties (μ_n) and (σ_n) are equivalent in the class of perfectly normal spaces.* □

Question 3.5. Does a perfectly normal space X satisfy property (σ_n), $n = 0, 1, 2, \ldots$, for an arbitrary $\mathcal{K}$?

Remark 3.6. The answer is "yes" if $\mathcal{K}$ contains a disconnected complex K. In fact, in this case, $\mathcal{K}$-$\operatorname{Ind} X = \operatorname{Ind} X$ (look at [4]) for every normal space X, and dimension Ind satisfies the countable sum theorem in the class of all perfectly normal spaces (look at [1,2]).

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